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Tri-bimaximal Neutrino Mixing from Orbifolding

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Abstract

We show that the A_4 discrete symmetry that naturally leads to tri-bimaximal neutrino mixing can be simply obtained as a result of an orbifolding starting from a model in 6 dimensions. This particular orbifolding has four fixed points where 4 dimensional branes are located and the tetrahedral symmetry of A_4 connects these branes. In this approach A_4 appears after the reduction from six to four dimensions as a remnant of the 6D space-time symmetry. A previously discussed supersymmetric version of A_4 is reinterpreted along these lines.

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1 Introduction

It is an experimental fact [1] that within measurement errors the observed neutrino mixing matrix is compatible with the so called tri-bimaximal form, introduced by Harrison, Perkins and Scott (HPS) [2]. It is an interesting challenge to formulate dynamical principles that, in a completely natural way, can lead to this specific mixing pattern as a first approximation, with small corrections determined by higher order terms in a well defined expansion. In a series of papers [3, 4] it has been pointed out that a broken flavour symmetry based on the discrete group A_4 appears to be particularly fit for this purpose. Other solutions based on continuous flavour groups like $SU(3)$ or $SO(3)$ have also been recently presented [5, 6], but the A_4 models have a very economical and attractive structure (for example, in terms of field content). A crucial feature of all HPS models is the mechanism used to guarantee the necessary VEV alignment of the flavon field φ_T which determines the charged lepton mass matrix with respect to the direction in flavour space chosen by the flavon φ_S that gives the neutrino mass matrix. In recent papers [7, 8] we have constructed explicit versions of A_4 model where the alignment problem is solved. In ref. [7] we adopted an extra dimensional framework, with φ_T and φ_S on different branes so that the minimization of the respective potentials is kept to a large extent independent. In ref. [8], we presented an alternative, perhaps more conventional, formulation of the A_4 model in 4 dimensions with supersymmetry (SUSY) at the price of introducing a somewhat less economic set of fields. Versions either with see-saw or without see-saw can be constructed. The existence of different realizations shows that the connection of A_4 with the HPS matrix is robust and does not necessarily require extra dimensions.

Another important aspect of the problem is that of trying to understand the dynamical origin of A_4 . As a first move in this direction, in ref. [8] we have reformulated A_4 as a subgroup of the modular group which often plays a role in the formalism of string theories, for example in the context of duality transformations [9]. In the present note we show that the A_4 symmetry can be simply obtained by orbifolding starting with a model in 6 dimensions (6D). In this approach A_4 appears as the remnant of the reduction from 6D to 4D space-time symmetry induced by the special orbifolding adopted. There are 4D branes at the four fixed points of the orbifolding and the tetrahedral symmetry of A_4 connects these branes. The standard model fields have components on the fixed point branes while the scalar fields necessary for the A_4 breaking are in the bulk.

In this paper, starting from a 6D field theory, we first introduce the specific orbifolding with four fixed points on which the 4D standard model fields live (while a number of additional gauge singlets are in the bulk) and specify how the A_4 transformations relate the field components on different branes or on the bulk. We then study the invariant interactions, local in 6D, constructed out of the fields in the theory which are invariant under A_4 . Finally we rederive the SUSY model for tri-bimaximal neutrino mixing in this particular framework.

2 A_4 as the isometry of T^2/Z_2

We consider a quantum field theory in 6 dimensions, with two extra dimensions compactified on an orbifold T^2/Z_2 . We denote by $z = x_5 + ix_6$ the complex coordinate describing the extra space. The torus T^2 is defined by identifying in the complex plane the points related by

$$\begin{aligned} z &\rightarrow z + 1 \\ z &\rightarrow z + \gamma \end{aligned} \quad \gamma = e^{i\frac{\pi}{3}} \quad , \quad (1)$$

where our length unit, $2\pi R$, has been set to 1 for the time being. The parity Z_2 is defined by

$$z \rightarrow -z \quad (2)$$

and the orbifold T^2/Z_2 can be represented by the fundamental region given by the triangle with vertices $0, 1, \gamma$, see Fig. 1. The orbifold has four fixed points, $(z_1, z_2, z_3, z_4) = (1/2, (1 + \gamma)/2, \gamma/2, 0)$. The fixed point z_4 is also represented by the vertices 1 and γ . In the orbifold, the segments labelled by a in Fig. 1, $(0, 1/2)$ and $(1, 1/2)$, are identified and similarly for those labelled by b , $(1, (1 + \gamma)/2)$ and $(\gamma, (1 + \gamma)/2)$, and those labelled by c , $(0, \gamma/2)$, $(\gamma, \gamma/2)$. Therefore the orbifold is a regular tetrahedron with vertices at the four fixed points. The symmetry of the uncompactified 6D space time is broken by

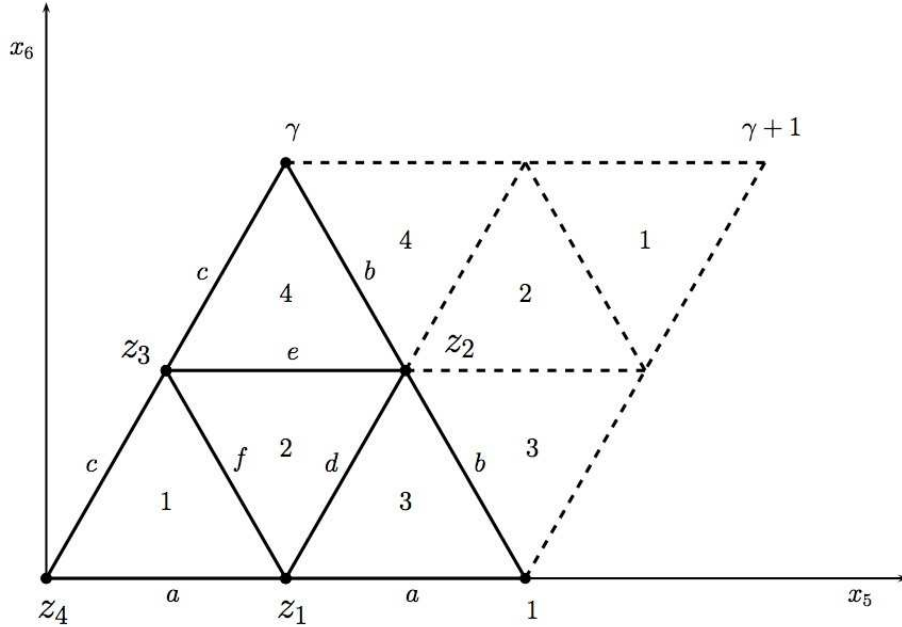


Figure 1: Orbifold T_2/Z_2 . The regions with the same numbers are identified with each other. The four triangles bounded by solid lines form the fundamental region, where also the edges with the same letters are identified. The orbifold T_2/Z_2 is exactly a regular tetrahedron with 6 edges a, b, c, d, e, f and four vertices z_1, z_2, z_3, z_4 , corresponding to the four fixed points of the orbifold.

compactification. Here we assume that, before compactification, the space-time symmetry

coincides with the product of 6D translations and 6D proper Lorentz transformations. The compactification breaks part of this symmetry. However, due to the special geometry of our orbifold, a discrete subgroup of rotations and translations in the extra space is left unbroken. This group can be generated by two transformations:

$$\begin{aligned}\mathcal{S}: \quad z &\rightarrow z + \frac{1}{2} \\ \mathcal{T}: \quad z &\rightarrow \omega z \quad \quad \omega \equiv \gamma^2 \quad .\end{aligned}\tag{3}$$

Indeed \mathcal{S} and \mathcal{T} induce even permutations of the four fixed points:

$$\begin{aligned}\mathcal{S}: \quad (z_1, z_2, z_3, z_4) &\rightarrow (z_4, z_3, z_2, z_1) \\ \mathcal{T}: \quad (z_1, z_2, z_3, z_4) &\rightarrow (z_2, z_3, z_1, z_4)\end{aligned}\quad ,\tag{4}$$

thus generating the group A_4 .¹ From the previous equations we immediately verify that \mathcal{S} and \mathcal{T} satisfy the characteristic relations obeyed by the generators of A_4 :

$$\mathcal{S}^2 = \mathcal{T}^3 = (\mathcal{ST})^3 = 1 \quad .\tag{6}$$

These relations are actually satisfied not only at the fixed points, but on the whole orbifold, as can be easily checked from the general definitions of \mathcal{S} and \mathcal{T} in eq. (3), with the help of the orbifold defining rules in eqs. (1) and (2). In our model the discrete group A_4 , together with 4D translations and 4D proper Lorentz transformations, can be seen as the subgroup of the space-time symmetry in six dimensions that survives compactification. In a similar context, the compactification of two extra dimensions on an orbifold T^2/Z_3 and its relation to the flavour symmetry Z_3 has been analyzed in ref. ([10]).

It is useful to represent the action of \mathcal{S} and \mathcal{T} on the fixed points by means of the four by four matrices S and T^{-1} respectively.

$$S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad T^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .\tag{7}$$

The matrices S and T satisfy the relations (6), thus providing a representation of A_4 . Since the only irreducible representations of A_4 are a triplet and three singlets, the 4D representation described by S and T is not irreducible. It decomposes into the sum of the invariant singlet plus the triplet representation. If we denote by $u = (u_1, u_2, u_3, u_4)^t$ (the suffix t denotes transposition) a multiplet transforming as

$$u \rightarrow Su \quad , \quad u \rightarrow Tu \quad ,\tag{8}$$

¹Notice that an odd permutation of the four fixed points can be generated by the parity:

$$z \rightarrow z^* \quad ,\tag{5}$$

that maps (z_1, z_2, z_3, z_4) into (z_1, z_3, z_2, z_4) and belongs to the full 6D Poincaré group, which, beyond 6D translations and proper Lorentz transformations, also includes discrete symmetries. Therefore, had we assumed 6D Poincaré as starting point in the uncompactified theory, we would have ended up with the product of 4D Poincaré times the discrete group S_4 .

under S and T respectively, then singlet corresponds to

$$u_1 = u_2 = u_3 = u_4 \quad , \quad (9)$$

while the triplet is obtained by imposing the constraint

$$\sum_{i=1}^4 u_i = 0 \quad . \quad (10)$$

Both conditions (9) and (10) are invariant under A_4 . To better visualize this decomposition, we consider the unitary matrix U given by:

$$U = \frac{1}{2} \begin{pmatrix} +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & -1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \end{pmatrix} . \quad (11)$$

This matrix maps S and T into matrices that are block-diagonal:

$$USU^\dagger = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & S_3 \end{array} \right) , \quad UTU^\dagger = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & T_3 \end{array} \right) , \quad (12)$$

where S_3 and T_3 are the generators of the three-dimensional representation:

$$S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad T_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} . \quad (13)$$

If $u = (u_1, u_2, u_3, u_4)^t$ transforms as specified in eq. (8), then $v \equiv (v_0, v_1, v_2, v_3)^t = Uu$ transforms as

$$v \rightarrow (USU^\dagger)v \quad , \quad v \rightarrow (UTU^\dagger)v \quad , \quad (14)$$

respectively. Therefore, if we parametrize u as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} v_0 \\ v_0 \\ v_0 \\ v_0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -v_1 + v_2 + v_3 \\ +v_1 - v_2 + v_3 \\ +v_1 + v_2 - v_3 \\ -v_1 - v_2 - v_3 \end{pmatrix} , \quad (15)$$

the components $(v_1, v_2, v_3)^t$ transform with S_3 and T_3 , whereas the component v_0 is left invariant by A_4 . It is useful to observe that v_0 is given by $v_0 = (u_1 + u_2 + u_3 + u_4)/2$ while the sum of all components of the last multiplet in eq. (15) vanishes, in agreement with the conditions (9) and (10). Finally, if we restrict to the case of a pure triplet by taking $v_0 = 0$, then v_1, v_2 and v_3 are given by:

$$\begin{pmatrix} 0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = U \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ -u_1 - u_2 - u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ u_2 + u_3 \\ u_1 + u_3 \\ u_1 + u_2 \end{pmatrix} . \quad (16)$$

3 Local interactions invariant under A_4

In this section we collect the rules to construct an A_4 invariant field theory in the 6D space-time $\mathcal{M} \times T^2/Z_2$. The fields of this theory can be either 4D fields living at the fixed points, in short ‘brane’ fields, or ‘bulk’ fields depending on both the uncompactified coordinates x and the complex coordinate z . The new essential feature with respect to a 4D formalism is that in general all particles have components over all four fixed points. Locality in 6D implies that at each fixed point only products of components on that brane are allowed in the interaction terms. This constraint reduces the number of invariant interactions that can be constructed out of brane fields. We now discuss the structure of the invariants in this context.

3.1 Brane fields

We first consider the case of brane fields and we denote by

$$a = (a_1(x), a_2(x), a_3(x), a_4(x)) \quad (17)$$

a set of fields localized at the fixed points (z_1, z_2, z_3, z_4) , respectively. For the time being we do not specify if a is a scalar, a spinor or a vector under the 4D Lorentz group. We denote by $\delta_i = \delta(z - z_i)$ the 2D Dirac deltas needed to construct an interaction term local in 6D, starting from brane fields. We observe that, if z undergoes the transformations (3), then the delta functions $\delta = (\delta_1, \delta_2, \delta_3, \delta_4)^t$ are mapped into ²

$$\begin{aligned} \mathcal{S} : \quad \delta &\rightarrow S\delta \\ \mathcal{T} : \quad \delta &\rightarrow T\delta \end{aligned} \quad , \quad (18)$$

where S and T are given in eq. (7). The A_4 transformations of a are naturally given by:

$$\begin{aligned} \mathcal{S} : \quad a &\rightarrow Sa \\ \mathcal{T} : \quad a &\rightarrow Ta \end{aligned} \quad , \quad (19)$$

According to our discussion in the previous section, the quadruplet a decomposes into a triplet plus the invariant singlet 1. If we introduce two such sets of brane fields, called a and b , transforming as specified in eq. (19), then it is easy to see that the only invariant under A_4 , bilinear in a and b and local in 6D is given by:

$$J^{(2)} = \sum_{i=1}^4 a_i b_i \delta_i \quad . \quad (20)$$

In particular, if $a = (a_c/2, a_c/2, a_c/2, a_c/2)$ and $b = (b_c/2, b_c/2, b_c/2, b_c/2)$ are two invariant singlets, then, after integrating over the z coordinate, the invariant $J^{(2)}$ is given by $\int d^2z J^{(2)} = a_c b_c$. If a is a singlet and b is a triplet, $J^{(2)}$ vanishes after integration over z ,

²Notice that the action of \mathcal{T} on the Dirac deltas is described by T , the inverse of the matrix T^{-1} that permutes the four fixed points, eq. (7).

because of eq. (10). If a and b are two triplets transforming as in eq. (19), they can be parametrized as shown in eq. (15):

$$a = \frac{1}{2} \begin{pmatrix} -v_1 + v_2 + v_3 \\ +v_1 - v_2 + v_3 \\ +v_1 + v_2 - v_3 \\ -v_1 - v_2 - v_3 \end{pmatrix}, \quad b = \frac{1}{2} \begin{pmatrix} -w_1 + w_2 + w_3 \\ +w_1 - w_2 + w_3 \\ +w_1 + w_2 - w_3 \\ -w_1 - w_2 - w_3 \end{pmatrix}. \quad (21)$$

In this case, after integration over z , the bilinear $J^{(2)}$ reads:

$$\int d^2 z J^{(2)} = v_1 w_1 + v_2 w_2 + v_3 w_3, \quad (22)$$

which is the familiar expression of the invariant under A_4 contained in the product of two triplet representations [7].

Locality in 6D provides some limitations in the construction of interaction terms. For instance, it will be important for the following discussion to note that if a and b are two triplets transforming as in (19), then it is not possible to construct a term bilinear in a and b , local in 6D and transforming as a $1'$ or a $1''$. This is easily seen by starting from the local bilinear

$$J' = \sum_{i=1}^4 y_i a_i b_i \delta_i, \quad (23)$$

where y_i are constants to be determined by imposing that J' transforms as a $1'$. In fact it is trivial to see that only the trivial solution $y_i = 0$ is allowed. This is because \mathcal{S} imposes $y_4 = y_1$ and $y_3 = y_2$; while \mathcal{T} requires $y_4 = \omega y_1$, hence $y_4 = y_1 = 0$, and $y_1 = \omega y_2 = \omega^2 y_2$, so that also $y_2 = y_3 = 0$. The same argument also shows that it is equally impossible to obtain $1''$.

To obtain a non-invariant singlet from two triplets one has two possibilities. The first one is to exploit bulk fields, as we shall see in detail in the next subsection. The second one is to make use of a freedom associated to the A_4 algebra, by generalizing the transformation properties of the brane fields in the following way:

$$\begin{aligned} \mathcal{S}: \quad a &\rightarrow Sa \\ \mathcal{T}: \quad a &\rightarrow \omega^{r_a} Ta \end{aligned}, \quad (24)$$

where ω is a cubic root of unity, eq. (3), and $r_a = (0, \pm 1)$.

Clearly these new transformations satisfy the A_4 algebra, eq. (6). The only difference with respect to the transformations in eq. (19) is in the phase factor ω^{r_a} . It is possible to show that, once the delta function transformations are specified as in eq. (18), then eq. (24) provides the only allowed generalization of eq. (19). If we call $\mathcal{R}_{0,-1,+1}$ these representations, we see that they are all reducible: \mathcal{R}_0 decomposes into a triplet plus the invariant singlet 1, \mathcal{R}_{+1} decomposes into a triplet plus the singlet $1'$ and \mathcal{R}_{-1} decomposes into a triplet plus the singlet $1''$. It is immediate to see that $J^{(2)}$ is left invariant by A_4 only if (a, b) transform as $\mathcal{R}_a, \mathcal{R}_b$ with $a + b = 0$. To build a non-invariant singlet one has to assign (a, b) to $(\mathcal{R}_0, \mathcal{R}_{\pm 1})$. For example, consider the case $\mathcal{R} = +1$ for b . Then the

triplet (w_1, w_2, w_3) can be embedded in b in the following way:

$$b = \frac{1}{2} \begin{pmatrix} -w_1 + \omega w_2 + \omega^2 w_3 \\ +w_1 - \omega w_2 + \omega^2 w_3 \\ +w_1 + \omega w_2 - \omega^2 w_3 \\ -w_1 - \omega w_2 - \omega^2 w_3 \end{pmatrix} . \quad (25)$$

Now the bilinear

$$\sum_{i=1}^4 a_i b_i \delta_i \quad , \quad (26)$$

is invariant under \mathcal{S} and picks up a phase ω under \mathcal{T} , that is it transforms as a singlet $1'$. After integrating over the coordinate z , we find

$$\int d^2 z \sum_{i=1}^4 a_i b_i \delta_i = v_1 w_1 + \omega v_2 w_2 + \omega^2 v_3 w_3 \quad . \quad (27)$$

This example shows that, although from the point of view of the group A_4 the triplet representations contained in \mathcal{R}_0 , \mathcal{R}_{+1} , \mathcal{R}_{-1} are all equivalent (they can be seen as the result of the multiplication of a triplet by the singlets 1 , $1'$, $1''$, respectively), in this 6D framework their difference is not irrelevant when building up local interactions covariant under A_4 .

Generalizing what done above, a local invariant $J^{(N)}$ of degree N , built out of M brane multiplets $a^{(I)}$ ($I = 1, \dots, M$) transforming as \mathcal{R}_{r_I} is given by:

$$J^{(N)} = \sum_{i=1}^4 (a_i^{(1)})^{n_1} \dots (a_i^{(M)})^{n_M} \delta_i \quad , \quad (28)$$

where $\sum_{I=1}^M n_I = N$ and $\sum_{I=1}^M r_I = 0 \pmod{3}$.

3.2 Bulk and brane fields

Here we consider the coupling between a bulk multiplet $\mathbf{B}(z) = (\mathbf{B}_1(z), \mathbf{B}_2(z), \mathbf{B}_3(z))$, transforming as a triplet of A_4 , and a brane multiplet $a = (a_1, a_2, a_3, a_4)$, transforming as \mathcal{R}_0 under A_4 . The dependence on the 4D space-time coordinates x is not made explicit in our notation. For the time being, we assume that the three components $\mathbf{B}_I(z)$ ($I = 1, 2, 3$) are scalars in 6D. The transformations of \mathbf{B} under A_4 are specified by:

$$\begin{aligned} \mathcal{S} : \quad \mathbf{B}'(z_S) &= S_3 \mathbf{B}(z) \quad z_S = z + \frac{1}{2} \\ \mathcal{T} : \quad \mathbf{B}'(z_T) &= T_3 \mathbf{B}(z) \quad z_T = \omega z \end{aligned} . \quad (29)$$

We write the most general local term bilinear in a and \mathbf{B} as:

$$J = \sum_{iK} \alpha_{iK} a_i \mathbf{B}_K(z) \delta_i \quad , \quad (30)$$

where α_{iK} is a four by three matrix of constant coefficients. It is not difficult to see that, in order to have J invariant under A_4 , we should choose

$$\alpha_{iK} = \frac{1}{2} \begin{pmatrix} -1 & +1 & +1 \\ +1 & -1 & +1 \\ +1 & +1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \quad , \quad (31)$$

up to an overall constant. If the brane multiplet a is a \mathcal{R}_0 triplet under A_4 , parametrized as in eq. (21), by choosing α_{iK} as in (31), after integration over z we get:

$$\begin{aligned} J = & \frac{1}{4}(-v_1 + v_2 + v_3)(-\mathbf{B}_1(z_1) + \mathbf{B}_2(z_1) + \mathbf{B}_3(z_1)) + \\ & + \frac{1}{4}(+v_1 - v_2 + v_3)(+\mathbf{B}_1(z_2) - \mathbf{B}_2(z_2) + \mathbf{B}_3(z_2)) \\ & + \frac{1}{4}(+v_1 + v_2 - v_3)(+\mathbf{B}_1(z_3) + \mathbf{B}_2(z_3) - \mathbf{B}_3(z_3)) \\ & + \frac{1}{4}(+v_1 + v_2 + v_3)(+\mathbf{B}_1(z_4) + \mathbf{B}_2(z_4) + \mathbf{B}_3(z_4)) \end{aligned} \quad (32)$$

If the triplet $\mathbf{B}(z)$ acquires a constant VEV $\langle \mathbf{B}(z) \rangle = (\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)$, essentially the only case that will be relevant for the discussion in the next session, then the invariant J becomes

$$J = v_1 \mathbf{B}_1 + v_2 \mathbf{B}_2 + v_3 \mathbf{B}_3 \quad . \quad (33)$$

Similarly, by requiring that J' given by

$$J' = \sum_{iK} \alpha'_{iK} a_i \mathbf{B}_K(z) \delta_i \quad , \quad (34)$$

transforms as a $1'$, we find that the matrix α'_{iK} should be given by

$$\alpha'_{iK} = \frac{1}{2} \begin{pmatrix} -1 & +\omega & +\omega^2 \\ +1 & -\omega & +\omega^2 \\ +1 & +\omega & -\omega^2 \\ -1 & -\omega & -\omega^2 \end{pmatrix} \quad . \quad (35)$$

In this case, after integration over z and after substitution of the triplet $\mathbf{B}(z)$ with its constant VEV, the quantity J of eq. (30) becomes

$$J = v_1 \mathbf{B}_1 + \omega v_2 \mathbf{B}_2 + \omega^2 v_3 \mathbf{B}_3 \quad . \quad (36)$$

Finally, the singlet $1''$ is obtained from J' , by substituting α'_{iK} with its complex conjugate α''_{iK} .

4 Orbifold realization of the A_4 model

Let's start by recalling the basic formulae for the baseline A_4 model for lepton masses and mixings in 4D with supersymmetry [8]. The full superpotential of the model is

$$w = w_l + w_d \quad (37)$$

where w_l is the term responsible for the Yukawa interactions in the lepton sector and w_d is the term responsible for the vacuum alignment. We now detail the structure of both in succession. The term w_l is given by

$$w_l = y_e e^c (\varphi_T l) + y_\mu \mu^c (\varphi_T l)'' + y_\tau \tau^c (\varphi_T l)' + (x_a \xi + \tilde{x}_a \tilde{\xi})(ll) + x_b (\varphi_S ll) + h.c. + \dots \quad (38)$$

To keep our formulae compact, we omit to write the Higgs fields $h_{u,d}$ and the cut-off scale Λ . For instance $y_e e^c(\varphi_T l)$ stands for $y_e e^c(\varphi_T l) h_d / \Lambda$, $x_a \xi(l l)$ stands for $x_a \xi(l h_u l h_u) / \Lambda^2$ and so on. The superpotential w_l contains the lowest order operators in an expansion in powers of $1/\Lambda$. Dots stand for higher dimensional operators. The “driving” term w_d reads:

$$w_d = M(\varphi_0^T \varphi_T) + g(\varphi_0^T \varphi_T \varphi_T) + g_1(\varphi_0^S \varphi_S \varphi_S) + g_2 \tilde{\xi}(\varphi_0^S \varphi_S) + g_3 \xi_0(\varphi_S \varphi_S) + g_4 \xi_0 \xi^2 + g_5 \xi_0 \xi \tilde{\xi} + g_6 \xi_0 \tilde{\xi}^2 \quad , \quad (39)$$

where φ_0^T , φ_0^S and ξ_0 are driving fields that allow to build a non-trivial scalar potential in the symmetry breaking sector. The superpotential w is invariant not only with respect to the gauge symmetry $SU(2) \times U(1)$ and the flavour symmetry A_4 , but also under a discrete Z_3 symmetry and a continuous $U(1)_R$ symmetry under which the fields transform as shown in the following table.

Field	1	e^c	μ^c	τ^c	$h_{u,d}$	φ_T	φ_S	ξ	$\tilde{\xi}$	φ_0^T	φ_0^S	ξ_0
A_4	3	1	1'	1''	1	3	3	1	1	3	3	1
Z_3	ω	ω^2	ω^2	ω^2	1	1	ω	ω	ω	1	ω	ω
$U(1)_R$	1	1	1	1	0	0	0	0	0	2	2	2

Table 1: Fields and their transformation properties under A_4 , Z_3 and $U(1)_R$.

We now show how this model can be derived from the 6D field theory with orbifolding. We start from an $N = 1$ chiral supersymmetric 6D field theory, corresponding to $N = 2$ SUSY in the 4D language. Such an extended SUSY is broken down to $N = 1$ SUSY by the Z_2 parity in the usual way. The lagrangian of the theory is the sum of a bulk term, depending on bulk fields and invariant under $N = 2$ SUSY, plus boundary terms localized at the four fixed points and invariant under the less restrictive $N = 1$ SUSY. Moreover at the fixed points we are allowed to localize brane $N = 1$ multiplets. In particular we choose as brane fields the gauge bosons of the SM gauge group, the SM fermions and two Higgs doublets h_u and h_d , together with their $N = 1$ superpartners. The remaining fields, namely the flavons and the driving fields are introduced as bulk hypermultiplets. In this way we avoid 6D gauge anomalies. Due to the orbifolding, out of the two $N = 1$ chiral supermultiplets contained in the generic bulk hypermultiplet only one possesses a zero mode. Here we are interested in the brane interactions of this particular multiplet, and we will use for it the $N = 1$ notation.

The dictionary between the 4D realization, specified by the superpotential w_l and the present 6D version, is given in table 2. We have denoted by l_i the lepton doublet supermultiplets, which are A_4 -triplet brane fields parametrized as in eq. (21):

$$l = \frac{1}{2} \begin{pmatrix} -l_e + l_\mu + l_\tau \\ +l_e - l_\mu + l_\tau \\ +l_e + l_\mu - l_\tau \\ -l_e - l_\mu - l_\tau \end{pmatrix} \quad . \quad (40)$$

4D	6D
$\xi(ll)$	$\sum_{i=1}^4 l_i l_i \xi(z) \delta_i$
$(\varphi_S ll)$	$\sum_{i=1}^4 l_i l_i \alpha_{iK} \varphi_{SK}(z) \delta_i$
$e^c(\varphi_T l)$	$\sum_{i=1}^4 e^c l_i \alpha_{iK} \varphi_{TK}(z) \delta_i$
$\mu^c(\varphi_T l)''$	$\sum_{i=1}^4 \mu^c l_i \alpha_{iK}'' \varphi_{TK}(z) \delta_i$
$\tau^c(\varphi_T l)'$	$\sum_{i=1}^4 \tau^c l_i \alpha_{iK}' \varphi_{TK}(z) \delta_i$

Table 2: Realization of 4D superpotential terms for w_l in terms of local 6D A_4 invariants. The 4D terms are obtained from the 6D ones by integrating over the complex coordinate z and by assuming a constant background value for the bulk multiplets $\langle \varphi_{S,T}(z) \rangle = \langle \varphi_{S,T} \rangle / \sqrt{V}$, $\langle \xi(z) \rangle = \langle \xi \rangle / \sqrt{V}$.

The charged leptons e^c , μ^c and τ^c are brane fields, having the same value at each fixed point. As anticipated, the flavon fields $\varphi_S(z)$, $\varphi_T(z)$ and $\xi(z)$ are bulk fields, depending on the extra coordinate z . In particular $\varphi_S(z)$ and $\varphi_T(z)$ are A_4 triplets, transforming as in eq. (29), while $\xi(z)$ is an A_4 invariant: $\xi'(z + 1/2) = \xi(z)$ and $\xi'(\omega z) = \xi(z)$. Each 4D superpotential term is reproduced, up to an overall constant, from the corresponding 6D term of the dictionary by integrating over the complex coordinate z and by assuming a constant, that is z -independent, background value for the bulk supermultiplets $\varphi_S(z)$, $\varphi_T(z)$ and $\xi(z)$. This last requirement is justified by the fact that we only need to discuss the expansion of w around the VEVs of the flavon fields. Barring a peculiar behaviour of such VEVs, we will look for minima of the scalar potential that do not depend on z and in our final expressions the bulk fields will be replaced by their constant VEVs. In this way the superpotential w_l is completely reproduced.

To correctly establish the relation between the 6D superpotential and the 4D one we should also pay attention to the overall normalization of w_l . The 6D superpotential w_l is linear in the bulk fields having mass dimension two and therefore carries an extra factor $1/\Lambda$ with respect to the 4D superpotential. Moreover, the VEV of the generic bulk field \mathbf{B} can be parametrized as $\langle B \rangle / \sqrt{V}$ where $\langle B \rangle$ is the VEV of the zero mode, of mass dimension one, and V is the volume of the extra compact space. Therefore, after spontaneous breaking of the A_4 symmetry, each bulk field \mathbf{B} enters the superpotential in

the dimensionless combination $\langle B \rangle / (\Lambda^2 \sqrt{V})$. Higher dimensional operators are suppressed by extra powers of this combination. To avoid large corrections to the HPS mixing scheme, such a combination is required to be at most of order λ^2 , $\lambda \approx 0.22$ being the Cabibbo angle. This is of no concern for the lepton sector of the theory, but it can be a potential problem for the extension of the A_4 model, both in its 4D and 6D realizations, to the quark sector. Indeed we expect that the mass of the top quark arises from an unsuppressed renormalizable operator, whereas a naive extension of the A_4 assignment to the quark sector of our 6D model would lead to a top mass depleted by an overall factor $\langle \varphi_T \rangle / (\Lambda^2 \sqrt{V})$ (with respect, say, to the W mass), which as we have seen is expected to be of order λ^2 .

Finally we need a similar dictionary for the driving part of the superpotential. It is easy to see that each 4D term in w_d can be reproduced starting from a corresponding 6D term, by assuming constant field configurations and by integrating over the coordinate z . The new feature when analyzing w_d is that in general there is no one-to-one correspondence between 4D and 6D terms as was the case for w_l because the number of local 6D invariants we can build from bulk fields is larger than the number of 4D invariants we have in w_d . This is not an obstacle in deriving the 4D theory. Since we are interested in constant field configurations of the flavon and driving fields, after integration over z our 6D driving superpotential will indeed give rise to the most general set of A_4 invariants in 4D. The result is nothing but the superpotential w_d given in eq. (39). At this point the discussion of the vacuum alignment proceeds exactly as in the 4D case, detailed in ref. [8].

The scalar potential is minimum at:

$$\begin{aligned} \langle \varphi_T \rangle &= \frac{1}{\sqrt{V}}(v_T, v_T, v_T) & \frac{v_T}{\Lambda \sqrt{V}} &= -\frac{M}{g} \\ \langle \varphi_S \rangle &= \frac{1}{\sqrt{V}}(v_S, 0, 0) & v_S^2 &= -\frac{g_4}{g_3} u^2 \\ \langle \xi \rangle &= \frac{1}{\sqrt{V}} u & u &\text{undetermined} \\ \langle \tilde{\xi} \rangle &= 0 & & \end{aligned} \quad (41)$$

At the leading order of the $1/\Lambda$ expansion, the mass matrix m_l for charged leptons is given by:

$$m_l = v_d \frac{v_T}{\Lambda^2 \sqrt{V}} \begin{pmatrix} y_e & y_e & y_e \\ y_\mu & y_\mu \omega & y_\mu \omega^2 \\ y_\tau & y_\tau \omega^2 & y_\tau \omega \end{pmatrix}, \quad (42)$$

and charged fermion masses are given by:

$$m_e = \sqrt{3} y_e v_d \frac{v_T}{\Lambda^2 \sqrt{V}}, \quad m_\mu = \sqrt{3} y_\mu v_d \frac{v_T}{\Lambda^2 \sqrt{V}}, \quad m_\tau = \sqrt{3} y_\tau v_d \frac{v_T}{\Lambda^2 \sqrt{V}}. \quad (43)$$

We can easily obtain a natural hierarchy among m_e , m_μ and m_τ by introducing an additional $U(1)_F$ flavour symmetry under which only the right-handed lepton sector is charged. In the flavour basis the neutrino mass matrix reads :

$$m_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} a + 2d/3 & -d/3 & -d/3 \\ -d/3 & 2d/3 & a - d/3 \\ -d/3 & a - d/3 & 2d/3 \end{pmatrix}, \quad (44)$$

where

$$a \equiv x_a \frac{u}{\Lambda^2 \sqrt{V}} \quad , \quad d \equiv x_d \frac{v_S}{\Lambda^2 \sqrt{V}} \quad , \quad (45)$$

and is diagonalized by the transformation:

$$U^T m_\nu U = \frac{v_u^2}{\Lambda} \text{diag}(a + d, a, -a + d) \quad , \quad (46)$$

with

$$U = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & +1/\sqrt{2} \end{pmatrix} \quad . \quad (47)$$

For the neutrino masses we obtain:

$$\begin{aligned} |m_1|^2 &= \left[-r + \frac{1}{8 \cos^2 \Delta (1 - 2r)} \right] \Delta m_{atm}^2 \\ |m_2|^2 &= \frac{1}{8 \cos^2 \Delta (1 - 2r)} \Delta m_{atm}^2 \\ |m_3|^2 &= \left[1 - r + \frac{1}{8 \cos^2 \Delta (1 - 2r)} \right] \Delta m_{atm}^2 \quad , \end{aligned} \quad (48)$$

where $r \equiv \Delta m_{sol}^2 / \Delta m_{atm}^2 \equiv (|m_2|^2 - |m_1|^2) / |m_3|^2 - |m_1|^2$, $\Delta m_{atm}^2 \equiv |m_3|^2 - |m_1|^2$ and Δ is the phase difference between the complex numbers a and d . For $\cos \Delta = -1$, we have a neutrino spectrum close to hierarchical:

$$|m_3| \approx 0.053 \text{ eV} \quad , \quad |m_1| \approx |m_2| \approx 0.017 \text{ eV} \quad . \quad (49)$$

In this case the sum of neutrino masses is about 0.087 eV. If $\cos \Delta$ is accidentally small, the neutrino spectrum becomes degenerate. The value of $|m_{ee}|$, the parameter characterizing the violation of total lepton number in neutrinoless double beta decay, is given by:

$$|m_{ee}|^2 = \left[-\frac{1 + 4r}{9} + \frac{1}{8 \cos^2 \Delta (1 - 2r)} \right] \Delta m_{atm}^2 \quad . \quad (50)$$

For $\cos \Delta = -1$ we get $|m_{ee}| \approx 0.005 \text{ eV}$, at the upper edge of the range allowed for normal hierarchy, but unfortunately too small to be detected in a near future. Independently from the value of the unknown phase Δ we get the relation:

$$|m_3|^2 = |m_{ee}|^2 + \frac{10}{9} \Delta m_{atm}^2 \left(1 - \frac{r}{2} \right) \quad , \quad (51)$$

which is a prediction of our model. In Fig. 2 we have plotted the neutrino masses predicted by the model.

In summary, we have obtained the baseline 4D A_4 model starting from a 6D realization, where all SM supermultiplets live at the fixed points of a T^2/Z_2 orbifold and the flavon and driving fields live in the bulk.

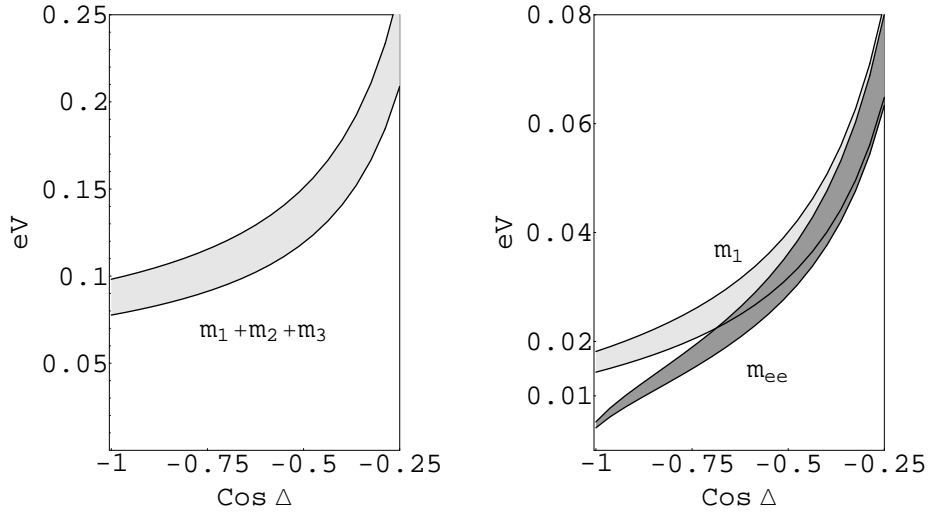


Figure 2: On the left panel, sum of neutrino masses versus $\cos \Delta$, the phase difference between a and b . On the right panel, the lightest neutrino mass, m_1 and the mass combination m_{ee} versus $\cos \Delta$. To evaluate the masses, the parameters $|a|$ and $|b|$ have been expressed in terms of $r \equiv \Delta m_{sol}^2 / \Delta m_{atm}^2 \equiv (|m_2|^2 - |m_1|^2) / (|m_3|^2 - |m_1|^2)$ and $\Delta m_{atm}^2 \equiv |m_3|^2 - |m_1|^2$. The bands have been obtained by varying Δm_{atm}^2 in its 3σ experimental range, $0.0020 \text{ eV} \div 0.0032 \text{ eV}$. There is a negligible sensitivity to the variations of r within its current 3σ experimental range, and we have realized the plots by choosing $r = 0.03$.

5 Conclusion

We have shown that extra dimensional theories with orbifolding provide a natural framework to interpret flavour symmetries as discrete permutational symmetries among fixed point branes. In particular, starting from a 6D theory, we have discussed an orbifolding with 4 fixed points leading to the A_4 flavour symmetry. In this picture A_4 together with 4D translations and 4D proper Lorentz transformations represents the subgroup of 6D space-time symmetry which is left unbroken in the theory after orbifolding and after locating the SM particles on the fixed point branes. Note that A_4 and not the full permutation group S_4 is the residual symmetry group because only even permutations can be seen as the result of a rigid space rotation. Each brane field, either a triplet or a singlet, has components on all of the four fixed points (in particular all components are equal for a singlet) but the interactions are local, i.e. all vertices involve products of field components at the same space-time point. This approach suggests a deep relation between flavour symmetry in 4D and space-time symmetry in extra dimensions. We have also demonstrated that a SUSY model of neutrino tri-bimaximal mixing based on A_4 , which we have formulated in a recent work [8], can be directly reinterpreted in the orbifolding approach.

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